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Bifurcation in gravity waves

By M. S. LONGUET-HIGGINS

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge, England, *and* Institute of Oceanographic Sciences, Wormley, Godalming, Surrey

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A new method is proposed for the calculation of gravity waves on deep water. This is based on some recently discovered quadratic identities between the Fourier coefficients a_n in Stokes's expansion. The identities are shown to be derivable from a cubic potential function, which in turn is related to the Lagrangian of the motion. A criterion for the bifurcation of uniform waves into a series of steady waves of non-uniform amplitude is expressed by the vanishing of a particular determinant with elements which are linear combinations of the coefficients a_n. The critical value of the wave steepness for the symmetric bifurcations discovered by Chen & Saffman (1980) are verified. It is shown that a truncated scheme consisting of only the coefficients a_0, a_1 and a_2 already exhibits Class 2 bifurcation, and similarly for Class 3. Asymmetric bifurcations are also discussed. A recent suggestion by Tanaka (1983) that gravity waves exhibit a Class 1 bifurcation at the point of maximum energy is shown to be incorrect.

1. Introduction

In an interesting paper, Chen & Saffman (1980) provided convincing numerical evidence that steady, irrotational waves in water of infinite depth are not unique. At a certain wave steepness the regular system of waves, of uniform amplitude, were shown to bifurcate into other series of steady waves of non-uniform amplitude. For example, in waves of Class 2, every second wave can be higher than the rest. In class n waves, the overall horizontal periodicity is n wavelengths of the uniform waves. Chen & Saffman also found that there were no Class 1 bifurcations, that is to say there were no steady waves having the same overall periodicity as the original series, but of a different surface profile.

However, Tanaka (1983) has suggested that Chen & Saffman's conclusions regarding the absence of a Class 1 bifurcation may not be correct. Tanaka had calculated the normal modes of instability of regular waves and found numerically that the frequency σ of the perturbation (in a frame of reference travelling with the unperturbed wave) vanished at a point very close to the steepness ak for which the energy density E was a maximum. (This occurs at a steepness ak less than $(ak)_{\max}$.) Tanaka suggested that the vanishing of σ implied a Class 1 bifurcation at that point, in conflict with Chen & Saffman's findings.

Now Chen & Saffman's method of calculation was by an integral equation involving the surface displacement, and this does not easily reveal the physical reasons for their remarkable findings. The purpose of the present paper is to look at the problem from a simpler point of view, namely the Fourier expansions employed originally by Stokes (1880). The idea is to exploit some simple, quadratic relations between the coefficients a_n in Stokes' expansion, which were found recently by the present author (Longuet-Higgins 1978).

As a first step we show, in §2, that these quadratic relations are all derivable from a potential function F which is cubic polynomial in the coefficients a_n. Not surprisingly, the function F can be shown (§3) to be closely related to the Lagrangian L of the wave motion (where L is the kinetic energy density minus the potential energy density). More usefully, the quadratic relations lead to a very simple scheme for the determination of the Fourier coefficients in terms of either the first coefficient a_0, or of any other convenient parameter μ along the branch of the solution curve, as described in §5. By this method one can determine the coefficients quickly and economically, up to a wave height within 1 percent of the maximum. This is without the use of Padé approximants or other methods for accelerating the convergence.

The analysis also yields a straightforward criterion for bifurcation of types 2 and 3 (and generally n) in terms of the Fourier coefficients a_n, and it is verified that Chen & Saffman's calculations of the critical wave steepness for Class 2 and Class 3 bifurcations are accurate to four decimal places.

A further advantage of the analysis is that by truncating the number of harmonics at only 2 or 3, a simplified model is obtained which itself exhibits the essential property of bifurcation (see §§6 and 7). The approach via Fourier coefficients is therefore both revealing and robust.

In §8 the analysis is extended to include asymmetric solutions, and criteria are given for this more general type of bifurcation, in terms of the a_t. By applying these criteria it is shown that if a Class 1 bifurcation takes place at the energy maximum $E = E_{\max}$, then it can only take the form of a pure phase shift. This supports Chen & Saffman's calculations, and throws doubt on the validity of Tanaka's conclusion.

2. Symmetric waves: Fourier coefficients

Consider a steady, progressive, irrotational wave travelling horizontally with speed c relative to deep water. Let us choose axes OX, OY moving with the wave, with OY vertically upwards and the origin O chosen so the mean surface level is $Y = -c^2/2g$. We assume that the motion is periodic in the X-direction with period 2π, the wavelength being a submultiple of 2π. If Φ denotes the velocity potential, then after Stokes (1880) we may express the coordinates (X, Y) in the form

$$(Y - \mathrm{i}X) - (\Psi - \mathrm{i}\Phi)/c = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \,\mathrm{e}^{\mathrm{i}n(\Phi+\mathrm{i}\Psi)/c}, \tag{2.1}$$

in which, for symmetric waves, the coefficients a_n are all real.

As shown elsewhere (Longuet-Higgins 1978, 1984*a*) the condition that the pressure be constant at the free surface ($\Psi = 0$) is equivalent to the set of relations

$$\int_{\Phi=0}^{2\pi/c} Y\,\mathrm{e}^{-\mathrm{i}n\Phi/c}\,(\mathrm{d}X + \mathrm{i}\,\mathrm{d}Y) = \begin{Bmatrix} -\pi c^2 & (n=0) \\ 0 & (n = 1, 2, \ldots) \end{Bmatrix} \tag{2.2}$$

and this is immediately seen to give rise to the relations

$$\left.\begin{aligned} a_0 + a_1 a_1 + 2a_2 a_2 + 3a_3 a_3 + \ldots &= -c^2, \\ a_1 + a_0 a_1 + 2a_1 a_2 + 3a_2 a_3 + \ldots &= 0, \\ a_2 + a_1 a_1 + 2a_0 a_2 + 3a_1 a_3 + \ldots &= 0, \\ a_3 + a_2 a_1 + 2a_1 a_2 + 3a_0 a_3 + \ldots &= 0. \\ \vdots\qquad & \end{aligned}\right\} \tag{2.3}$$

between the coefficients a_n and the wave-speed c^2. It will be noticed that these are quadratic at most, in comparison with the usual cubic system which follows from a direct application of Bernoulli's equation at the free surface.

Moreover it may be verified that (2.3) can also be expressed compactly in the form

$$\frac{\partial F}{\partial a_n} = 0, \quad n = 0, 1, 2, \ldots, \tag{2.4}$$

where
$$F = (\alpha + J + a_0 K) + \tfrac{1}{4}(a_0 + c^2)^2, \tag{2.5}$$

and we have written

$$\begin{aligned} \alpha = a_1 \quad & (a_1 a_2 + a_2 a_3 + a_3 a_4 + \ldots) \\ & + 2a_2(a_1 a_3 + a_2 a_4 + a_3 a_5 + \ldots) \\ & + 3a_3(a_1 a_4 + a_2 a_5 + a_3 a_6 + \ldots) \\ & + \ldots . \end{aligned} \tag{2.6}$$

Also
$$\mathrm{J} = \tfrac{1}{2}(a_1^2 + \mathrm{a}_2^2 + \mathrm{a}_3^2 + \ldots), \tag{2.7}$$

$$K = \tfrac{1}{2}(a_1^2 + 2a_2^2 + 3a_3^2 + \ldots). \tag{2.8}$$

That is to say, for any given value of c^2 the possible values of $a_0, a_1, a_2, \ldots$ are those that correspond to *stationary* values of the function $F(a_0, a_1, a_0, \ldots)$. This gives rise to many useful properties of the system (2.3), and to a convenient method of computation, as shown below.

We note that the first of equations (2.3) may be written in the form

$$a_0 + 2K = -c^2. \tag{2.9}$$

Hence we have also
$$\frac{\partial F}{\partial c^2} = \tfrac{1}{2}(a_0 + c^2) = -K. \tag{2.10}$$

3. F related to the Lagrangian L

Some previously known relations will enable us to relate the function F to the Lagrangian density L defined as

$$L = T - V, \tag{3.1}$$

where T and V denote respectively the kinetic, and potential, energy densities. For it has been shown (Longuet-Higgins 1975) that for gravity waves in water of uniform depth

$$\mathrm{d}L = I\,\mathrm{d}c, \tag{3.2}$$

where I is the momentum density, and also that

$$I = cK, \tag{3.3}$$

where K is defined by (2.8) (see Longuet-Higgins 1984*a*). Hence

$$2\,\mathrm{d}L = K\,\mathrm{d}c^2. \tag{3.4}$$

On the other hand from (2.4) and (2.10) it follows that

$$\mathrm{d}F = \sum_i \frac{\partial F}{\partial a_i}\mathrm{d}a_i + \frac{\partial F}{\partial c^2}\mathrm{d}c^2 = -K\,\mathrm{d}c^2. \tag{3.5}$$

Hence
$$\mathrm{d}F = -2\mathrm{d}L. \tag{3.6}$$

But when $a_n = 0, n \geqslant 1$, then J, K, F and L all vanish together, hence on integrating (3.6) we have

$$F = -2L. \tag{3.7}$$

A more direct proof can be given as follows. In Longuet-Higgins (1984*a*) it is shown that equations (2.3) imply

$$3\alpha + 2(J + a_0 K) = 0. \tag{3.8}$$

So from (2.5), (2.9) and (3.8)

$$F = \tfrac{1}{3}(J + a_0 K) + K^2, \tag{3.9}$$

and on substituting for a_0 from (2.9) we have

$$3F = J - c^2 K + K^2. \tag{3.10}$$

This compares with the expression

$$6L = -J + c^2 K - K^2 \tag{3.11}$$

found in §6 of Longuet-Higgins (1984*a*).

4. Further differential relations

Let the system (2.3) be written as

$$F_i = 0, \quad i = 0, 1, 2, \ldots, \tag{4.1}$$

where $F_i \equiv \partial F / \partial a_i$. If $(a_0, a_1, a_2, \ldots; c^2)$ is any solution of these equations, then a neighbouring solution $(a_0 + \mathrm{d}a_0, a_1 + \mathrm{d}a_1, \ldots; c^2 + \mathrm{d}c^2)$ will satisfy

$$\mathrm{d}F_i = 0, \quad i = 0, 1, 2, \ldots, \tag{4.2}$$

that is

$$\sum_j \frac{\partial F_i}{\partial a_j} \mathrm{d}a_j = -\frac{\partial F_i}{\partial c^2} \mathrm{d}c^2. \tag{4.3}$$

The system may be displayed in the form

$$\begin{pmatrix} (1+1) & (a_1 + a_1) & (2a_2 + 2a_2) & (3a_3 + 3a_3) & \cdots \\ (a_1 + a_1) & (1 + a_0 + 2a_2) & (2a_1 + 3a_3) & (3a_2 + 4a_4) & \cdots \\ (2a_2 + 2a_2) & (2a_1 + 3a_3) & (1 + 2a_0 + 4a_4) & (3a_1 + 5a_5) & \cdots \\ (3a_3 + 3a_3) & (3a_2 + 4a_4) & (3a_1 + 5a_5) & (1 + 3a_0 + 6a_6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \times \begin{pmatrix} \tfrac{1}{2}\mathrm{d}a_0, \\ \mathrm{d}a_1 \\ \mathrm{d}a_2 \\ \mathrm{d}a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} -\mathrm{d}c^2 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{4.4}$$

The symmetry of the matrix on the left is guaranteed by the relations

$$\frac{\partial F_i}{\partial a_j} = \frac{\partial^2 F}{\partial a_i \partial a_j}. \tag{4.5}$$

On multiplying the first of equations (4.4) by a_0, the second by a_1, the third by a_2, etc. and adding, making use of (2.9), we obtain

$$\mathrm{d}J + a_0 \mathrm{d}K = 2K \, \mathrm{d}a_0. \tag{4.6}$$

For the reasons mentioned in § 1, we are interested in the particular points at which the phase speed is stationary with respect to increments in the other parameters, that is

$$\mathrm{d}c = 0; \quad \mathrm{d}L = 0. \tag{4.7}$$

By (4.4) the condition for this is clearly that the determinant

$$\Delta \equiv \begin{vmatrix} (1+1) & (a_1+a_1) & (2a_2+2a_2) & (3a_3+3a_3) & \dots \\ (a_1+a_1) & (1+a_0+2a_2) & (2a_1+3a_3) & (3a_2+4a_4) & \dots \\ (2a_2+2a_2) & (2a_1+3a_3) & (1+2a_0+4a_4) & (3a_1+5a_5) & \dots \\ (3a_3+3a_3) & (3a_2+4a_4) & (3a_1+5a_5) & (1+3a_0+6a_6) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} \tag{4.8}$$

shall vanish.

We are also interested in the points at which the total energy,

$$E = T + V, \tag{4.9}$$

is stationary. From the general relation

$$\mathrm{d}E = c\,\mathrm{d}I, \tag{4.10}$$

(which follows from (3.2)), we have that

$$\mathrm{d}E = 0, \quad \mathrm{d}I = 0 \tag{4.11}$$

simultaneously. A criterion can be stated in terms of the Fourier coefficients $a_0, a_1, a_2, \dots$. For then

$$\mathrm{d}(c^2K^2) = \mathrm{d}(I^2) = 2I\,\mathrm{d}I = 0. \tag{4.12}$$

Hence

$$K\,\mathrm{d}c^2 = -2c^2\,\mathrm{d}K = -(a_0+2K)\,\mathrm{d}(a_0+c^2), \tag{4.13}$$

that is

$$(a_0+3K)\,\mathrm{d}c^2 + (a_0+2K)\,\mathrm{d}a_0 = 0. \tag{4.14}$$

From (4.4) this implies that

$$\begin{vmatrix} (a_0+3K) & (2a_0+4K) & 0 & 0 & \dots \\ 1 & (1+1) & (a_1+a_1) & (2a_2+2a_2) & \dots \\ 0 & (a_1+a_1) & (1+a_0+2a_2) & (2a_1+3a_3) & \dots \\ 0 & (2a_2+2a_2) & (2a_1+3a_3) & (1+2a_0+4a_4) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} = 0. \tag{4.15}$$

K is of course defined in terms of the a_i by equation (2.8). The condition (4.15) may be written also as

$$(a_0+3K)\,\Delta = (2a_0+4K)\,\Delta', \tag{4.16}$$

where Δ' denotes the principal minor of Δ, that is to say

$$\Delta' \equiv \begin{vmatrix} (1+a_0+2a_2) & (2a_1+3a_3) & (3a_2+4a_4) & \dots \\ (2a_1+3a_3) & (1+2a_0+4a_4) & (3a_1+5a_5) & \dots \\ (3a_2+4a_4) & (3a_1+5a_5) & (1+3a_0+6a_6) & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix}. \tag{4.17}$$

5. Calculation of the coefficients

Equations (2.3) provide a simple means of calculating the coefficients a_i. Since the first equation is the only one involving the phase speed c, we may leave this until last, and work with the remaining equations. We have then to solve

$$\left.\begin{aligned} F_1 &\equiv a_1 + a_0a_1 + 2a_1a_2 + 3a_2a_3 + \dots = 0, \\ F_2 &\equiv a_2 + a_1a_1 + 2a_0a_2 + 3a_1a_3 + \dots = 0, \\ F_3 &\equiv a_3 + a_2a_1 + 2a_1a_2 + 3a_0a_3 + \dots = 0, \\ &\dots, \end{aligned}\right\} \tag{5.1}$$

as a function of some parameter μ taken along the curve. At first it is possible to take $\mu = a_0$. Then (5.1) are to be solved for $a_1, a_2, \ldots$, given the value of a_0.

We note that (5.1) may be expressed as

$$\frac{\partial G}{\partial a_i} = 0, \quad i = 1, 2, \ldots, \tag{5.2}$$

where G is the reduced potential

$$G \equiv \alpha + J + a_0 K, \tag{5.3}$$

in which the terms in c^2 and a_0 alone have been omitted (cf. 2.5).

Suppose that we are given, or can guess, an approximation $(a_1^{(1)}, a_2^{(1)}, \ldots)$ to the solution for a certain value of a_0. For example this might be the exact solution at a neighbouring point on the solution curve. Then to find a closer approximation we may (in general) calculate the corresponding values $F_i^{(1)}$ of F_i for each i, and solve the equations

$$\sum_j \frac{\partial F_i^{(1)}}{\partial a_j} \delta a_j = -F_i^{(1)} \tag{5.4}$$

for the increments δa_j, as in Newton's method. It is convenient to display this system in the form

$$\begin{pmatrix} (1+a_0+2a_1) & (2a_1+3a_3) & (3a_2+4a_4) & \cdots \\ (2a_1+3a_3) & (1+2a_0+4a_4) & (3a_1+5a_5) & \cdots \\ (3a_2+4a_4) & (3a_1+5a_5) & (1+3a_0+6a_6) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \times \begin{pmatrix} \delta a_1 \\ \delta a_2 \\ \delta a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} -F_1^{(1)} \\ -F_2^{(1)} \\ -F_3^{(1)} \\ \vdots \end{pmatrix}. \tag{5.5}$$

which also involves a symmetric matrix.

The method has the advantage that all the elements of the matrix are simple, linear combinations of the Fourier coefficients, and so easy to handle. No use is made of arbitrary power series expansions for each coefficient.

Figure 1 shows the values of the phase-speed, calculated in this way plotted (+) as a function of the wave steepness

$$ak = a_1 + a_3 + a_5 + \ldots \tag{5.6}$$

for Class 1 waves, that is to say those in which the first harmonic a_1 is dominant, so we may take the wavenumber k equal to 1. The total number N of harmonics did not exceed 320 (and generally was much less), and the convergence was not artificially accelerated. The method takes the calculation well past the steepness corresponding to the first energy maximum ($ak = 0.4292$) but not quite up to the first maximum of the phase speed ($ak = 0.4359$).

Evidently the method will fail if the parameter μ (in this case a_0) is locally stationary. This is characterized by the vanishing of the determinant Δ' of the matrix (5.5). In figure 2 we have plotted the ratio Δ'/Δ'', where Δ'' is the principal minor of Δ', and it can be seen that this goes to zero at about the point $ak = 0.434$, intermediate between the maxima of E and c. When plotted against ak in figure 3, it can be seen that $-a_0$ is approaching a maximum at this point. We may recall the maxima in the other coefficients $a_1, a_2, \ldots$ as shown in figure 3 of Schwartz (1974).

This general difficulty may be overcome by choosing a parameter μ that behaves monotonically throughout the complete range of wave steepness. Several such

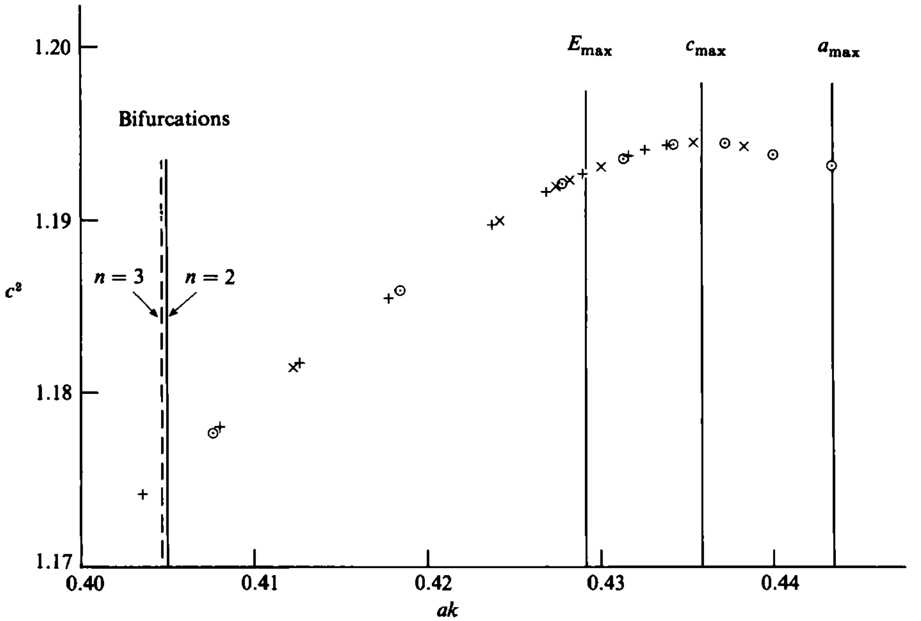

FIGURE 1. Graph of c^2 against ak for regular waves of Class 1, at the high values of the wave steepness. ⊙ Padé approximants (Longuet-Higgins 1975); + present paper, equations (5.5); × present paper, equations (5.10).

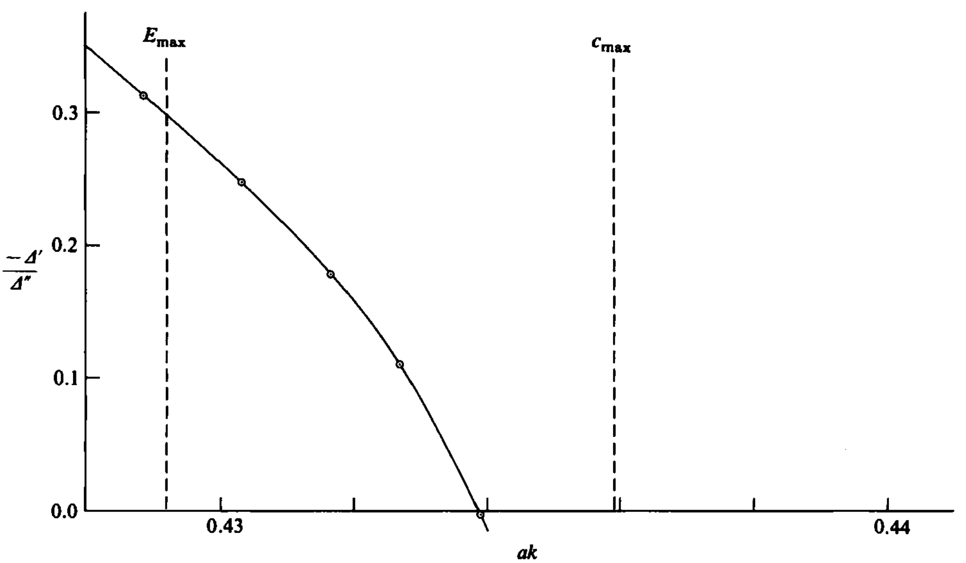

FIGURE 2. The ratio Δ'/Δ'' for the matrix of (5.5).

parameters have been employed by various authors. One is the crest-to-trough waveheight $2a$, which was introduced by Schwartz (1974), but this has an upper limit which is not known *a priori*. Longuet-Higgins (1975) and Cokelet (1977) used parameters which ranged monotonically from 0 to 1. Others have been introduced by Chen & Saffman (1980) and by Tanaka (1983).

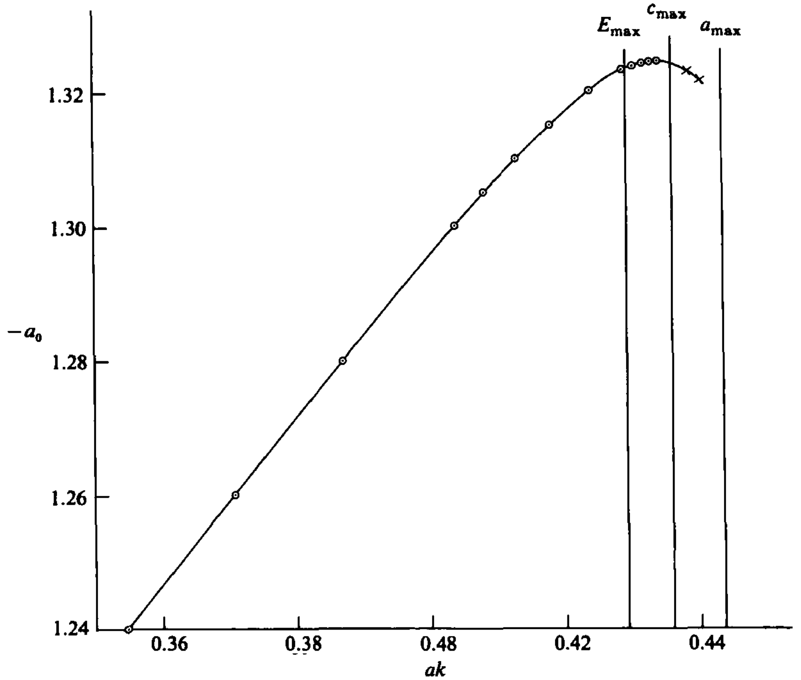

FIGURE 3. Graph of the lowest coefficient a_0 as a function of ak.

The simplest parameter, however, would seem to be

$$Q = 1-\tfrac{1}{2}q^2_{\text{crest}} \tag{5.7}$$

which for limiting waves tends to the known value 1, and which is expressed very simply and naturally in terms of the Fourier coefficients. For, by Bernoulli's theorem,

$$Q = 1+Y(0) = 1+\tfrac{1}{2}a_0+a_1+a_2+\ldots, \tag{5.8}$$

and it follows that

$$\mathrm{d}Q = \tfrac{1}{2}\mathrm{d}a_0+\mathrm{d}a_1+\mathrm{d}a_2+\ldots. \tag{5.9}$$

For a given value of Q, the equations to be employed in the successive approximation procedure are now

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ (a_1+a_1) & (1+a_0+2a_1) & (2a_1+3a_3) & (3a_2+4a_4) & \ldots \\ (2a_2+2a_2) & (2a_1+3a_3) & (1+2a_0+4a_4) & (3a_1+5a_5) & \ldots \\ (3a_3+3a_3) & (3a_2+4a_4) & (3a+5a_5) & (1+3a_0+6a_6) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \times \begin{pmatrix} \tfrac{1}{2}\delta a_0 \\ \delta a_1 \\ \delta a_2 \\ \delta a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} Q-Q^{(1)} \\ -F_1^{(1)} \\ -F_2^{(1)} \\ -F_3^{(1)} \\ \vdots \end{pmatrix}. \tag{5.10}$$

Some points computed by this method are shown by the crosses (×) in figure 1, and it will be seen that the method now takes the computation well past the point of maximum phase-speed and onto the descending part of the curve. As before, we took $N = 320$ and no Padé approximants were used.

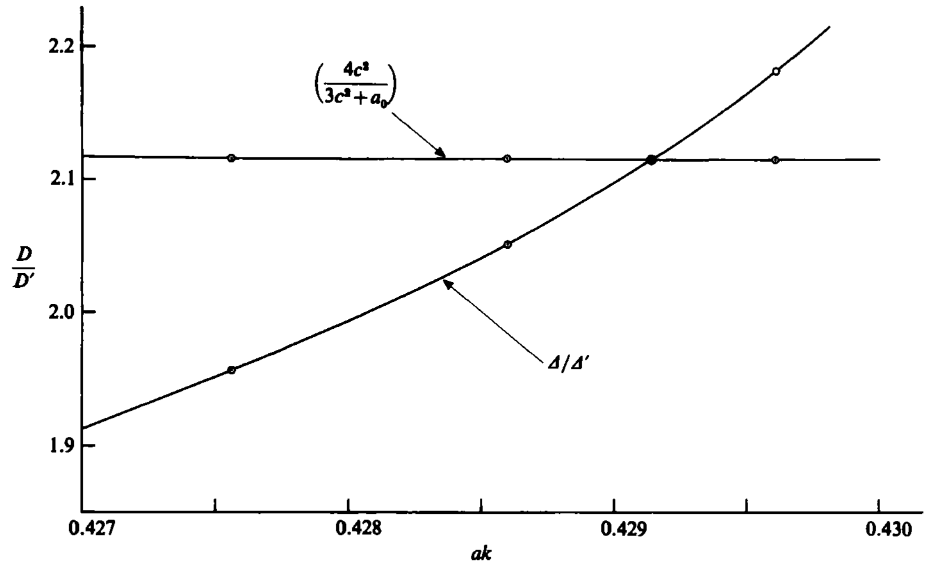

FIGURE 4. The ratio D/D' in equation (5.11), verifying the position of the energy maximum $E_{\max}$.

In addition we made a numerical check of the condition derived in §4 for a stationary value of E. Now (4.16) may be written

$$\frac{\Delta}{\Delta'} = \frac{2a_0+4K}{a_0+3K} = \frac{4c^2}{3c^2+a_0}. \tag{5.11}$$

In figure 4 we have plotted both the left- and right-hand sides of (5.11) as a function of ak. It will be seen that they do indeed intersect in the neighbourhood of $ak = 0.429$, the value of ak for which $E = E_{\max}$ (see Longuet-Higgins 1975; Longuet-Higgins & Fox 1978). The value of ak determined as in figure 4 was in fact 0.42914, with an uncertainty of two digits in the fifth decimal place.

6. Symmetric bifurcations: $n = 2$

In a wave of Class 2, where the second harmonic is dominant, the regular series of waves is characterized by the relations

$$\left.\begin{aligned} a_{2i+1} &= 0, \\ a_{2i} &= \tfrac{1}{2}a_i' \end{aligned}\right\} \quad i = 0, 1, 2, \ldots. \tag{6.1}$$

where a_i' is the corresponding coefficient in waves of Class 1. In such a series we obviously have in general $\mathrm{d}a_{2i+1} = 0$.

At a point of bifurcation, however, we look for a branch on which

$$\mathrm{d}a_{2i+1} \neq 0. \tag{6.2}$$

At the bifurcation point itself, where both (6.1) and (6.2) are satisfied, (5.5) will be seen to split into two independent subsystems, for the even and the odd increments respectively. That for the odd increments $\mathrm{d}a_{2i+1}$ can be written

$$\begin{pmatrix} (1+a_0+2a_2) & (3a_2+4a_4) & (5a_4+6a_6) & \cdots \\ (3a_2+4a_4) & (1+3a_0+6a_6) & (5a_2+8a_8) & \cdots \\ (5a_4+6a_6) & (5a_2+8a_3) & (1+5a_0+10a_{10}) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \times \begin{pmatrix} da_1 \\ da_3 \\ da_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{6.3}$$

Clearly this will have a non-zero solution only if the determinant of the system vanishes. We may express this in terms of the coefficients a_i' of the corresponding Class 1 waves by making the substitution

$$a_{2i} = \tfrac{1}{2}a_i' \tag{6.4}$$

After suppressing the primes $'$ (this need cause no confusion), we find as a criterion for this type of bifurcation

$$D \equiv \begin{vmatrix} (1+\tfrac{1}{2}a_0+a_1) & (\tfrac{3}{2}a_1+2a_2) & (\tfrac{5}{2}a_2+3a_3) & \cdots \\ (\tfrac{3}{2}a_1+2a_2) & (1+\tfrac{3}{2}a_0+3a_3) & (\tfrac{5}{2}a_1+4a_4) & \cdots \\ (\tfrac{5}{2}a_2+3a_3) & (\tfrac{5}{2}a_1+4a_4) & (1+\tfrac{5}{2}a_0+5a_5) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \tag{6.5}$$

We examined the condition (6.5) numerically, with a total number of harmonics N up to 320. In figure 5 are plotted some computed values of the ratio D/D', where D' is the principal minor of D (i.e. the determinant obtained by omitting the first row and the first column). It will be seen that this ratio behaves smoothly, and changes sign at about $ak = 0.405$. A precise value is

$$ak = 0.4049615 \tag{6.6}$$

corresponding to a height–length ratio $2a/\lambda = 0.128903$. This agrees to four figures with the value given by Chen & Saffman (1980).

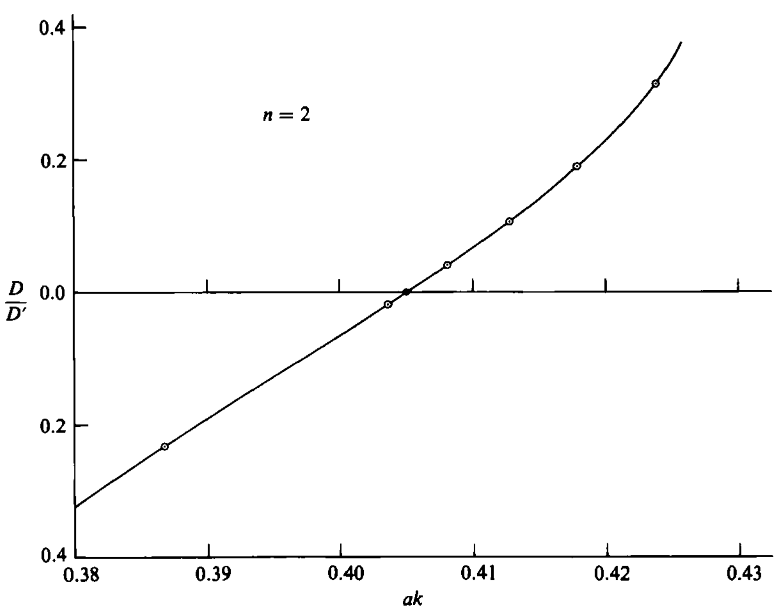

FIGURE 5. Graph of D/D' for equations (6.4).

The parameter b defined by Chen & Saffman (1980) is equivalent to

$$b = 1 - q_{\text{crest}}^2/c^2, \tag{6.7}$$

where q_{crest} denotes the particle speed at the crest in the stationary frame of reference: $q_{\text{crest}} = (\partial X/\partial \Phi)^{-1}_{\Phi=0}$. In terms of the Fourier coefficients a_i we have

$$b = 1 - (a_1 + 2a_2 + 3a_3 + \ldots)^{-2}. \tag{6.8}$$

This parameter was also computed and the critical value was found to be

$$b_c = 0.8797579, \tag{6.9}$$

which differs from that given by Chen & Saffman only in the fifth decimal place.

To throw some light on the reasons for the existence of this bifurcation, we may note that a similar bifurcation is to be found even in the drastically truncated system of equations in which only the coefficients a_0, a_1 and a_2 are retained. Then the potential function G of (5.3) reduces to

$$G = a_1^2 a_2 + \tfrac{1}{2}(a_1^2 + a_2^2) + \tfrac{1}{2}a_0(a_1^2 + 2a_2^2) \tag{6.10}$$

and (2.3) becomes

$$\left.\begin{aligned} a_1 + a_0 a_1 + 2a_1 a_2 &= 0, \\ a_2 + a_1 a_1 + 2a_0 a_2 &= 0, \end{aligned}\right\} \tag{6.11}$$

while the bifurcation condition (6.3) becomes simply

$$1 + a_0 + 2a_2 = 0. \tag{6.12}$$

So we have a bifurcation at

$$a_0 = -\tfrac{1}{2}, \quad a_1 = 0, \quad a_2 = -\tfrac{1}{4} \tag{6.13}$$

(see figure 6), the wave steepness being

$$ak = |2a_2| = 0.5 \tag{6.14}$$

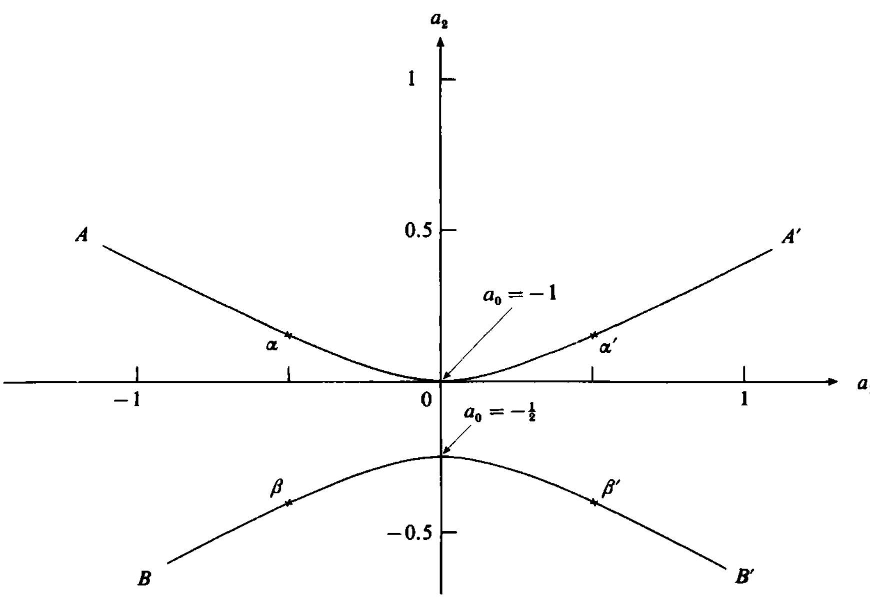

FIGURE 6. Class 2 bifurcation seen in solutions to the truncated system (6.11): AA and $0a_2$: uniform waves; BB': non-uniform waves.

(cf. the exact value 0.405). For general values of a_0, (6.11) and (6.12) have the parametric solution

$$\left.\begin{aligned} a_1 &= \pm[(a_0+\tfrac{1}{2})(a_0+1)]^{\frac{1}{2}}, \\ a_2 &= -\tfrac{1}{2}(a_0+1). \end{aligned}\right\} \tag{6.15}$$

Solutions with $a_0 \leqslant -1$ (figure 6, curve AA') correspond to regular waves; those with $a_0 \geqslant -\frac{1}{2}$ (curve BB') correspond to nonuniform waves. The solutions are restricted by the condition for limiting waves, which may be taken as

$$\tfrac{1}{2}a_0 \pm a_1 + a_2 = 0. \tag{6.16}$$

For each curve we find $a_0 = -(3+\sqrt{5})/4 = -1.3090$. These points are marked on the corresponding curves in figure 6 by the letters α, α'; β, β'.

This analysis suggests that the Class 2 bifurcation is a simple consequence of the interaction between the fundamental wave and its 2nd subharmonic.

It is interesting that figure 6 indicates that, on the curve BB' for irregular waves, the second harmonic a_2 is always positive; there is no analogous branch with $a_2 < 0$. This implies that irregular waves can exist in which the crests are symmetric fore-and-aft and the troughs are asymmetric; but not vice versa.

Nevertheless it would seem worthwhile to explore the branch on which, in place of (6.4), we have

$$a_{2i} = \tfrac{1}{2}a_i' \times (-1)^i. \tag{6.17}$$

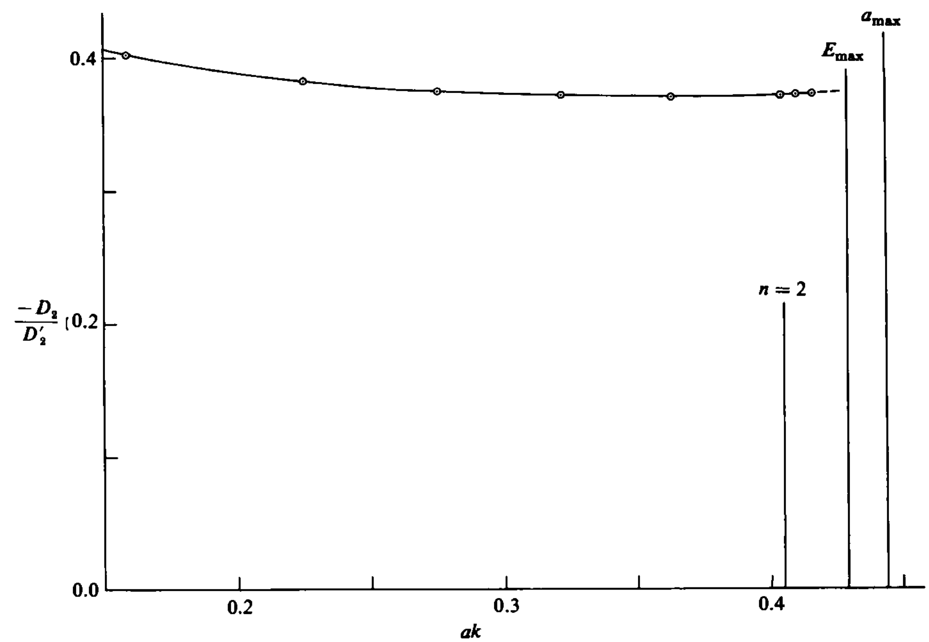

FIGURE 7. The ratio D_2/D_2' for the determinant (6.18).

This implies that we explore the behaviour of the determinant

$$D_2 \equiv \begin{vmatrix} (1+\frac{1}{2}a_0-a_1) & (-\frac{3}{2}a_1+2a_2) & (\frac{5}{2}a_2-3a_3) & \dots \\ (-\frac{3}{2}a_1+2a_2) & (1+\frac{3}{2}a_0-3a_3) & (-\frac{5}{2}a_1+4a_4) & \dots \\ (\frac{5}{2}a_2-3a_3) & (-\frac{5}{2}a_1+4a_4) & (1+\frac{5}{2}a_0-5a_5) & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix}. \tag{6.18}$$

Figure 7 shows the ratio D_2/D_2' for this determinant, and it can be seen that it remains almost a constant over the range of interest. We may conclude that there are no trough-symmetric Class 2 bifurcations over this range.

7. Symmetric bifurcations: $n = 3$

In a similar way, regular waves of Class 3, in which the third harmonic dominates, are characterized by

$$\left.\begin{aligned} a_{3i+1} = a_{3i+2} &= 0, \\ a_{3i} &= \tfrac{1}{3}a_i', \end{aligned}\right\} \quad i = 0, 1, 2, \ldots, \tag{7.1}$$

and if we seek another branch on which

$$\mathrm{d}a_{3i+1}, \quad \mathrm{d}a_{3i+2} \neq 0 \tag{7.2}$$

then for the bifurcation point itself we get two independent systems of equations for the $\mathrm{d}a_i$, of which one is

$$\left(\begin{array}{cc|cc|cc} (1+a_0) & 3a_3 & 4a_3 & 6a_6 & 7a_6 & 9a_9 \\ 3a_3 & (1+2a_0) & 6a_6 & 5a_3 & 9a_9 & 8a_6 \\ \hline 4a_3 & 6a_6 & (1+4a_0) & 9a_9 & 7a_3 & 12a_{12} \\ 6a_6 & 5a_3 & 9a_9 & (1+5a_0) & 12a_{12} & 8a_3 \\ \hline 7a_6 & 9a_9 & 7a_3 & 12a_{12} & (1+7a_0) & 15a_{15} \\ 9a_9 & 8a_6 & 12a_{12} & 8a_3 & 15a_{15} & (1+8a_0) \\ \hline \end{array}\right) \times \begin{pmatrix} \mathrm{d}a_1 \\ \mathrm{d}a_2 \\ \mathrm{d}a_4 \\ \mathrm{d}a_5 \\ \mathrm{d}a_7 \\ \mathrm{d}a_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{7.3}$$

Now writing $a_{3i} = \frac{1}{3}a_i'$, then $a_i' = a_i$ we obtain as the condition for Class 3 bifurcation that

$$D = \left|\begin{array}{cc|cc|cc|} (1+\frac{1}{3}a_0) & a_1 & \frac{4}{3}a_1 & 2a_2 & \frac{7}{3}a_2 & 3a_3 \\ a_1 & (1+\frac{2}{3}a_0) & 2a_2 & \frac{5}{3}a_1 & 3a_3 & \frac{8}{3}a_2 \\ \hline \frac{4}{3}a_1 & 2a_2 & (1+\frac{4}{3}a_0) & 3a_3 & \frac{7}{3}a_1 & 4a_4 \\ 2a_2 & \frac{5}{3}a_1 & 3a_3 & (1+\frac{5}{3}a_0) & 4a_4 & \frac{8}{3}a_1 \\ \hline \frac{7}{3}a_2 & 3a_3 & \frac{7}{3}a_1 & 4a_4 & (1+\frac{7}{3}a_0) & 5a_5 \\ 3a_3 & \frac{8}{3}a_2 & 4a_4 & \frac{8}{3}a_1 & 5a_5 & (1+\frac{8}{3}a_0) \\ \hline \end{array}\right| \tag{7.4}$$

should vanish. Again we calculated D/D' where $D^{(l)}$ denotes the determinant obtained from D by omitting the first l rows and columns. Since, however, D' changed sign close to D it was found preferable to plot the ratio D/D'', which varied more smoothly (see figure 8). From the graph it will be seen that D/D'' changes sign at nearly the same value as for Class 2 bifurcation, a precise value being

$$ak = 0.40469 \tag{7.5}$$

equivalent to $2a/\lambda = 0.12882$. This again agrees with Chen & Saffman's value (0.1288). However, for the critical value of the parameter b we find

$$b_{\mathrm{c}} = 0.87906, \tag{7.6}$$

differing from theirs in the fifth decimal place.

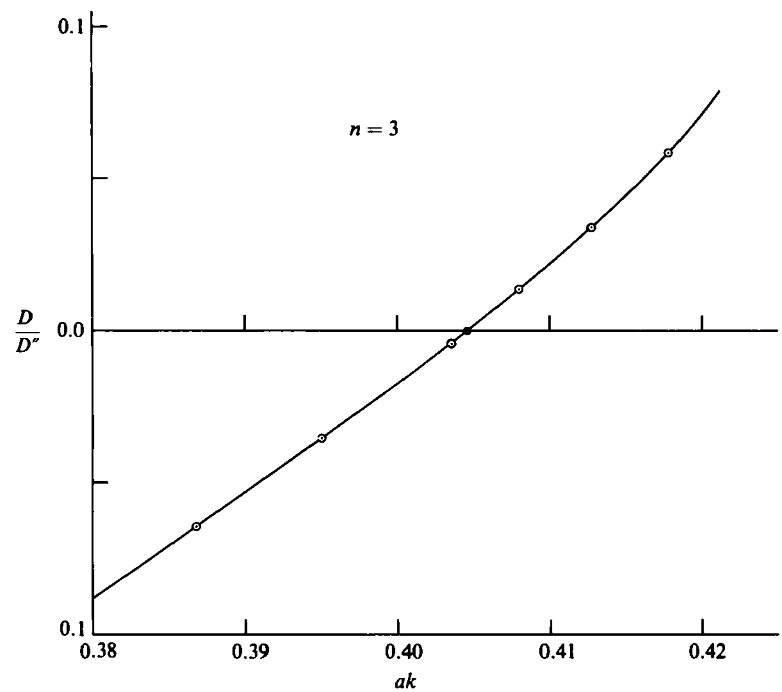

FIGURE 8. Graph of D/D'' for the Class 3 bifurcation conditions (7.4).

In the truncated system of equations, in which we set $a_i = 0$ for $i > 3$, we have

$$G = a_1^2 a_2 + 3a_1 a_2 a_3 + \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) + \tfrac{1}{2}a_0(a_1^2 + 2a_2^2 + 3a_3^2) \tag{7.7}$$

and in general we must solve

$$\left.\begin{aligned} a_1(1+a_0) + a_2(2a_1 + 3a_3) &= 0, \\ a_2(1+2a_0) + a_1(a_1 + 3a_3) &= 0, \\ a_3(1+3a_0) + 3a_1 a_2 &= 0. \end{aligned}\right\} \tag{7.8}$$

By eliminating first a_1 and then a_3 we may obtain the parametric solutions

$$\left.\begin{aligned} a_2 &= \tfrac{1}{9}[(1+3a_0) \pm \{(1+3a_0)(10+12a_0)\}^{\frac{1}{2}}], \\ a_1^2 &= \frac{(1+2a_0)(1+3a_0)}{\pm[(1+3a_0)(10+12a_0)]^{\frac{1}{2}}}, \\ a_3 &= -\frac{3a_1 a_2}{1+3a_0} \end{aligned}\right\} \tag{7.9}$$

These are sketched in figure 9. To help visualize the curves in three dimensions, each branch is joined to its projection on the 'horizontal' plane $a_2 = 0$. The branch AA' represents the regular solutions; in these a_2 is always positive. The branch BB' passes through the point $P = (a_1, a_2, a_3) = (0, 0, \sqrt{2}/9)$. This implies a critical wave steepness

$$ak = |3a_3| = \sqrt{2}/3 = 0.471\ldots. \tag{7.10}$$

Presumably this branch corresponds to the solutions traced by Chen & Saffman. A restriction on the validity of the solutions is imposed by the rough condition that

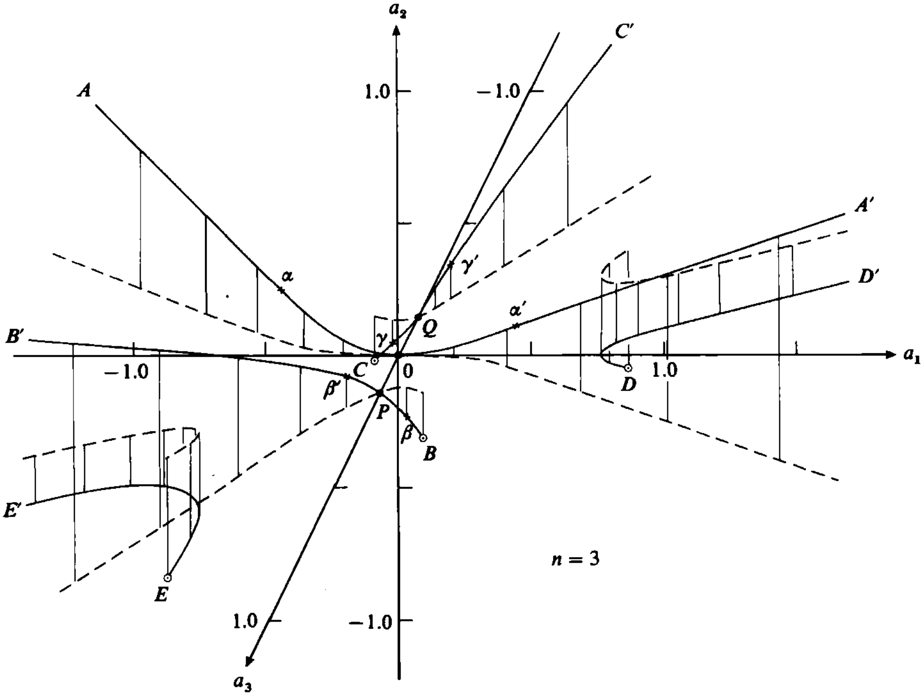

FIGURE 9. Class 3 bifurcation seen in solutions to the truncated system (7.6): AA' and Oa_2: uniform waves, BB' and CC': non-uniform waves.

$Y \leqslant 0$ at every point on the surface; otherwise, in the exact solution, q^2 would have to be negative. This implies in particular that

$$\tfrac{1}{2}a_0 + a_1 \cos\frac{n\pi}{3} + a_2 \cos\frac{2n\pi}{3} + a_3 \cos n\pi \leqslant 0, \tag{7.11}$$

where $n = 0, 1, 2$ and 3. By this criterion, allowable solutions on the curves AA' and BB' lie only between the marked crosses, α, α' and β, β'.

By the same criterion, the two branches DD' and EE' in figure 9 lie well outside the allowable range of parameters and presumably do not correspond to nontrivial solutions of the complete system of equations.

There remains the branch CC'. This, however, simply represents the same set of solutions as BB', but with the signs of a_1 and a_3 reversed (QC corresponds to PB, and QC' to PB'). This change implies a shift in phase of the previous solutions by π so that the point $X = 0$ now lies at a wave trough instead of at a crest. In this form, the curve CC' has a cusp at Q.

It may be noted that the point C in figure 8 corresponds to the solution

$$a_0 = a_1 = a_2 = a_3 = -\tfrac{1}{6}, \tag{7.12}$$

which, together with $a_i = 0$, $i > 3$, may be verified as a solution of the original equations (2.3). Generally, a solution of the truncated system, for any given n, can be seen to be

$$a_0 = a_1 = \ldots = a_n = -\frac{2}{n(n+1)}. \tag{7.13}$$

But clearly as $n \to \infty$ this solution tends to the trivial solution in which all the a_i vanish.

8. Asymmetric bifurcations

From §5 it is clear that a necessary condition for a symmetric bifurcation is that the determinant of (5.10) shall vanish. For regular waves of Class 1, we have seen that this condition appears not to be satisfied for regular waves of any steepness. We turn now to the question whether there could be any bifurcation of regular, Class 1 waves into *asymmetric* forms.

For this problem we need to generalize the equation of §2 by allowing all the coefficients a_n to be complex (except a_0). Thus in (2.1) we may take

$$a_n = p_n + \mathrm{i}q_n, \tag{8.1}$$

where p_n and q_n are real and $q_0 = 0$. It is convenient also to write

$$b_0 = 1, \quad b_n = a_n, \quad n = 1, 2, 3, \dots. \tag{8.2}$$

Equations (2.2) are still valid, but now the more general form of (2.3) is

$$\left.\begin{aligned} a_0 + a_1^* b_1 + a_2^* b_2 + a_3^* b_3 + \dots &= -c^2, \\ a_1 + a_0 b_1 + a_1^* b_2 + a_2^* b_3 + \dots &= 0, \\ a_2 + a_1 b_1 + a_0 b_2 + a_1^* b_3 + \dots &= 0, \\ a_3 + a_2 b_1 + a_1 b_2 + a_0 b_3 + \dots &= 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \end{aligned}\right\}. \tag{8.3}$$

Each of these equations, after the first, has both a real and imaginary component. Then the real and imaginary parts of (2.3) become respectively

$$\left.\begin{aligned} p_0 + (p_1 p_1 + q_1 q_1) + 2(p_2 p_2 + q_2 q_2) + 3(p_3 p_3 + q_3 q_3) + \dots &= -c^2, \\ p_1 + (p_0 p_1 \qquad\quad) + 2(p_1 p_2 + q_1 q_2) + 3(p_2 p_3 + q_2 q_3) + \dots &= 0, \\ p_2 + (p_1 p_1 - q_1 q_1) + 2(p_0 p_2 \qquad\quad) + 3(p_1 p_3 + q_1 q_3) + \dots &= 0, \\ p_3 + (p_2 p_1 - q_2 q_1) + 2(p_1 p_2 - q_1 q_2) + 3(p_0 p_3 \qquad\quad) + \dots &= 0, \\ \vdots \qquad\qquad & \end{aligned}\right\}, \tag{8.4}$$

and

$$\left.\begin{aligned} q_1 + (p_0 q_1 \qquad\quad) + 2(p_1 q_2 - p_2 q_1) + 3(p_2 q_3 - p_3 q_2) + \dots &= 0, \\ q_2 + (p_1 q_1 + p_1 q_1) + 2(p_0 q_2, \qquad\quad) + 3(p_1 q_3 - p_3 q_1) + \dots &= 0, \\ q_3 + (p_2 q_1 + p_1 q_2) + 2(p_1 q_2 + p_2 q_1) + 3(p_0 q_3 \qquad\quad) + \dots &= 0, \\ \vdots \qquad\qquad & \end{aligned}\right\}. \tag{8.5}$$

We see that the differentials

$$\left.\begin{aligned} \mathrm{d}\boldsymbol{p} &= (\tfrac{1}{2}p_0, \mathrm{d}p_1, \mathrm{d}p_2, \dots) \\ \mathrm{d}\boldsymbol{q} &= (\mathrm{d}q_1, \mathrm{d}q_2, \dots) \end{aligned}\right. \tag{8.6}$$

will then satisfy

$$\left(\begin{array}{c|c} \boldsymbol{P} & \boldsymbol{Q} \\ \hline \boldsymbol{Q}^T & \boldsymbol{R} \end{array}\right) \times \begin{pmatrix} \mathrm{d}\boldsymbol{p}^{\mathrm{T}} \\ \mathrm{d}\boldsymbol{q}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} -\mathrm{d}c^2 \\ \mathbf{0} \end{pmatrix} \tag{8.7}$$

where

$$\boldsymbol{P} = \begin{pmatrix} (1+1) & (p_1+p_1) & (2p_2+2p_2) & (3p_3+3p_3) & \\ (p_1+p_1) & (1+p_0+2p_2) & (2p_1+3p_3) & (3p_2+4p_4) & \cdots \\ (2p_2+2p_2) & (2p_1+3p_3) & (1+2p_0+4p_4) & (3p_1+5p_5) & \cdots \\ (3p_3+3p_3) & (3p_2+4p_4) & (3p_1+5p_5) & (1+3p_0+6p_6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{8.8}$$

$$\boldsymbol{Q} = \begin{pmatrix} (q_1+q_1) & (2q_2+2q_2) & (3q_3+3q_3) & \cdots \\ (\ \ 2q_2) & (2q_1+3q_3) & (3q_2+4q_4) & \cdots \\ (-2q_1+3q_3) & (\ \ 4q_4) & (3q_1+5q_5) & \cdots \\ (-3q_2+4q_4) & (-3q_1+5q_5) & (\ \ 6q_6) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{8.9}$$

and

$$\boldsymbol{R} = \begin{pmatrix} (1+p_0-2p_2) & (2p_1-3p_3) & (3p_2-4p_4) & \cdots \\ (2p_1-3p_3) & (1+2p_0-4p_4) & (3p_1-5p_5) & \cdots \\ (3p_2-4p_4) & (3p_1-5p_5) & (1+3p_0-6p_6) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{8.10}$$

It will be observed that the matrix $\boldsymbol{R}$ is similar to that of the principal minor of $\boldsymbol{P}$, except that in each element the sign of one of the terms is reversed.

To these equations must be added one for whatever parameter $\mu = \mu(p_0, p_1, p_2 \ldots; q_1, q_2 \ldots)$ is used along the branch in question, namely

$$\frac{\partial\mu}{\partial p_0}\mathrm{d}p_0 + \sum_{i=1}^{\infty}\left(\frac{\partial\mu}{\partial p_i}\mathrm{d}p_i + \frac{\partial\mu}{\partial q_i}\mathrm{d}q_i\right) = \mathrm{d}\lambda. \tag{8.11}$$

For bifurcation points lying on the regular branch we may take

$$p_i = a_i, \quad q_i = 0, \tag{8.12}$$

and it can be seen that the submatrix $\boldsymbol{Q}$ is null, and the system (8.6) breaks up into two independent sets of equations, one for determining the $\mathrm{d}p_i$, or their ratios, and the other for determining the $\mathrm{d}q_i$. That for determining the dq_i can be written

$$\boldsymbol{R} \times \mathrm{d}\boldsymbol{q} = \boldsymbol{0} \tag{8.13}$$

where now

$$\boldsymbol{R} = \begin{pmatrix} (1+a_0-2a_2) & (2a_1-3a_3) & (3a_2-4a_3) & \cdots \\ (2a_1-3a_3) & (1+2a_0-4a_4) & (3a_1-5a_5) & \cdots \\ (3a_2-4a_4) & (3a_1-5a_5) & (1+3a_0-6a_6) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{8.14}$$

One would expect the full (8.7) to be satisfied identically by an asymmetric solution corresponding to a pure phaseshift, that is to say

$$\mathrm{d}p_n = nq_n\,\mathrm{d}\theta, \quad \mathrm{d}q_n = -np_n\,\mathrm{d}\theta, \tag{8.15}$$

where $\mathrm{d}\theta$ is a small phase angle. Thus (8.13) should be satisfied by

$$\mathrm{d}q_n = np_n, \quad n = 0, 1, 2, \ldots. \tag{8.16}$$

This may be verified by direct substitution. One of (8.13) is therefore redundant, and it may be replaced by the condition that $q_1 \equiv 0$ or

$$\mathrm{d}q_1 = 0. \tag{8.17}$$

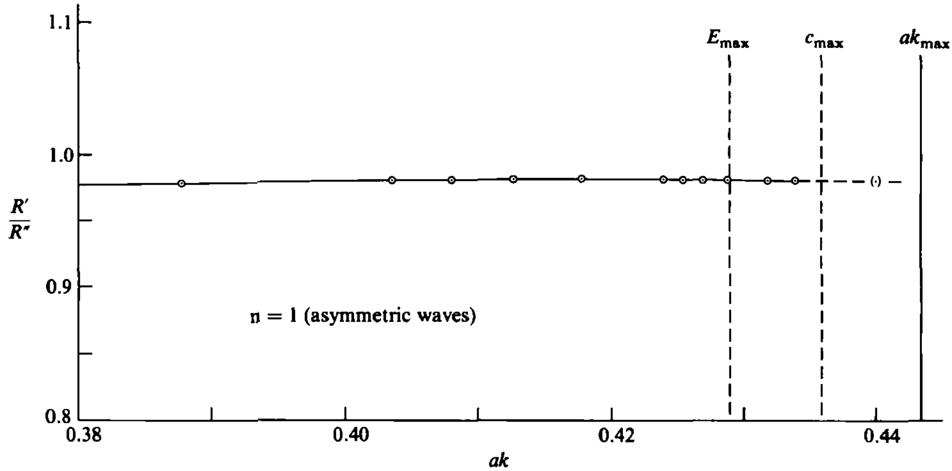

FIGURE 10. Graph of R'/R'' for the asymmetric bifurcation, given by (8.14).

This replaces the top row of R by the vector $(1,0,0,\ldots)$. Hence a necessary condition for the existence of non-zero solutions to (8.14), other than the phaseshift solution, is that the principal minor

$$R' = \begin{vmatrix} (1+2a_0-4a_4) & (3a_1-5a_3) & \ldots \\ (3a_1-5a_3) & (1+3a_0-6a_6) & \ldots \\ \vdots & \vdots & \end{vmatrix} = 0. \tag{8.18}$$

In figure 10 we have plotted some calculated values of the ratio R'/R'' where R'' denotes the principal minor of R'. It will be seen that this ratio is almost constant at around 0.98 over the whole of the range of interest. In particular it shows no sign whatever of varying significantly from this value in the neighbourhood of $E = E_{max}$ or $c = c_{max}$. The ratios R''/R''' and R''/R'''' also were nearly constant and of order 1. Lastly it was verified numerically that R/R' was always very small – of order 10^{-10} at most.

Thus it appears that the regular Class 1 waves have no asymmetric bifurcation, and hence no bifurcation at all, apart from a pure phase shift, throughout the range $0 < ak < 0.436$.

9. Discussion and conclusions

By exploiting the quadratic identities (5.1) between the Fourier coefficients a_i, we have been able to develop a method of calculation for Stokes waves of arbitrary steepness which avoids expansions in power series or the use of Padé approximants. We have shown that criteria for bifurcation and for other properties of the motion can be expressed in terms of the a_i. Not only does this method confirm in a simple way the critical wave steepness for wave bifurcation determined by earlier authors, but by truncating the Fourier series at, say, $i = 2$ or 3 it provides a simple model for understanding the existence of bifurcation points. In view of the relatively high wave steepnesses at which these phenomena occur, this was perhaps a surprise.

We have shown also that the bifurcation which was claimed by Tanaka (1983) to exist near the point $ak = 0.429$ can be no more than a pure phase shift. The relation betweeen bifurcations and normal-mode instabilities at zero frequency is examined in a companion paper (Longuet-Higgins 1984b).

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